



# Laplace's transform of fractional order via the Mittag–Leffler function and modified Riemann–Liouville derivative

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## ARTICLE INFO

### Article history:

Received 29 September 2008

Received in revised form 22 May 2009

Accepted 22 May 2009

### Keywords:

Fractional derivative

Fractional Taylor's series

Mittag–Leffler function

Fractional transform

Laplace's transform

## ABSTRACT

We propose a (new) definition of a fractional Laplace's transform, or Laplace's transform of fractional order, which applies to functions which are fractional differentiable but are not differentiable, in such a manner that they cannot be analyzed by using the Djrbashian fractional derivative. After a short survey on fractional analysis based on the modified Riemann–Liouville derivative, we define the fractional Laplace's transform. Evidence for the main properties of this fractal transformation is given, and we obtain a fractional Laplace inversion theorem.

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## 1. Introduction

The Laplace's transform  $L\{f(x)\} := F(s)$ ,  $s \in C$ , of a  $\Re \rightarrow C$  function  $f(x)$  is defined by the integral (the symbol  $:=$  means that the left side is defined by the right side)

$$L\{f(x)\} := F(s) := \int_0^{\infty} e^{sx} f(x) dx, \quad (1.1)$$

when it converges. It is very useful in applied mathematics, for instance for solving some differential equations and partial differential equations, and in automatic control, where it defines a transfer function. Its discrete counterpart provides the so-called generative function which is useful in the theory of finite difference equations.

The function  $f(x)$  so involved is usually continuous and continuously differentiable, and the question is what happens when it is continuous but with a fractional derivative of order  $\alpha$ ,  $0 < \alpha < 1$ , only. Two instances can occur. In the first one,  $f(x)$  has both a continuous derivative and a fractional derivative, in which case the expression (1.1) is quite meaningful. In the second case,  $f(x)$  has a derivative of order  $\alpha$ ,  $0 < \alpha < 1$ , but no derivative, and then (1.1) fails to apply; and as result we have to find an alternative.

The purpose of the present contribution is exactly to provide a possible approach to this alternative. The work is organized as follows. For the convenience of the reader, firstly we shall give a brief background on the definition of the fractional derivative as we use it (Section 2), then we shall define Laplace's transform of fractional order and we shall derive some of its main basic properties (Section 3). Then we shall compare the fractional Laplace's transform and the Laplace's transform, we shall state a convolution theorem (Section 4) and we shall obtain the inversion formula (Section 5).

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## 2. Background on fractional derivatives (revisited)

### 2.1. Fractional derivatives via fractional difference

**Definition 2.1.** Let  $f : \Re \rightarrow \Re, x \rightarrow f(x)$  denote a continuous (but not necessarily differentiable) function, and let  $h > 0$  denote a constant discretization span. Define the forward operator  $FW(h)$  by the equality (the symbol  $:=$  means that the left side is defined by the right side)

$$FW(h)f(x) := f(x+h); \quad (2.1)$$

then the fractional difference of order  $\alpha$ ,  $0 < \alpha < 1$ , of  $f(x)$  is defined by the expression [1–7]

$$\Delta^\alpha f(x) := (FW - 1)^\alpha = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f[x + (\alpha - k)h], \quad (2.2)$$

and its fractional derivative of order  $\alpha$  is defined by the limit

$$f^{(\alpha)}(x) = \lim_{h \downarrow 0} \frac{\Delta^\alpha f(x)}{h^\alpha}. \quad (2.3)$$

### 2.2. Modified fractional Riemann–Liouville derivative (via an integral)

*An alternative to the Riemann–Liouville definition of the fractional derivative*

In order to circumvent some drawbacks involved in the classical Riemann–Liouville definition, we have proposed the following alternative referred to as the modified Riemann–Liouville derivative [3]:

**Definition 2.2** (Riemann–Liouville Definition Revisited). Refer to the function  $f(x)$  of Definition 2.1.

(i) Assume that  $f(x)$  is a constant  $K$ . Then its fractional derivative of order  $\alpha$  is

$$D_x^\alpha K = K \Gamma^{-1}(1 - \alpha) x^{-\alpha}, \quad \alpha \leq 0, \quad (2.4)$$

$$= 0, \quad \alpha > 0. \quad (2.5)$$

(ii) When  $f(x)$  is not a constant, then we will set

$$f(x) = f(0) + (f(x) - f(0)),$$

and its fractional derivative will be defined by the expression

$$f^{(\alpha)}(x) = D_x^\alpha f(0) + D_x^\alpha (f(x) - f(0)),$$

in which, for negative  $\alpha$ , one has

$$D_x^\alpha (f(x) - f(0)) := \frac{1}{\Gamma(-\alpha)} \int_0^x (x - \xi)^{-\alpha-1} f(\xi) d\xi, \quad \alpha < 0, \quad (2.6)$$

whilst for positive  $\alpha$ , we will set

$$D_x^\alpha (f(x) - f(0)) = D_x^\alpha f(x) = D_x (f^{(\alpha-1)}(x)). \quad (2.7)$$

When  $n \leq \alpha < n + 1$ , we will set

$$f^{(\alpha)}(x) := (f^{(\alpha-n)}(x))^{(n)}, \quad n \leq \alpha < n + 1, n \geq 1. \quad (2.8)$$

We shall refer to this fractional derivative as the *modified Riemann Liouville derivative*, and it is in order to point out that it is strictly equivalent to Definition 2.1, via Eq. (2.2).

### 2.3. Taylor's series of fractional order

**Proposition 2.1.** Assume that the continuous function  $f : \Re \rightarrow \Re, x \rightarrow f(x)$  has a fractional derivative of order  $k\alpha$ , for any positive integer  $k$  and any  $\alpha$ ,  $0 < \alpha \leq 1$ ; then the following equality holds, which is (with the notation  $\Gamma(1 + \alpha k) =: (\alpha k)!$ ) [1–3]

$$f(x+h) = \sum_{k=0}^{\infty} \frac{h^{\alpha k}}{(\alpha k)!} f^{(\alpha k)}(x), \quad 0 < \alpha \leq 1, \quad (2.9)$$

with the notation  $D^{2\alpha} f(x) := D^\alpha D^\alpha f(x)$ .

## 2.4. Integration with respect to $(dx)^\alpha$

The integral with respect to  $(dx)^\alpha$  is defined as the solution of the fractional differential equation

$$dy = f(x)(dx)^\alpha, \quad x \geq 0, y(0) = 0, \quad (2.10)$$

which is provided by the following result:

**Lemma 2.1.** Let  $f(x)$  denote a continuous function; then the solution  $y(x)$ ,  $y(0) = 0$ , of Eq. (2.10) is defined by the equality

$$\begin{aligned} y &= \int_0^x f(\xi)(d\xi)^\alpha \\ &= \alpha \int_0^x (x - \xi)^{\alpha-1} f(\xi) d\xi, \quad 0 < \alpha < 1. \end{aligned} \quad (2.11)$$

To some extent, these results are more or less related to those of Kolwankar [8,9]. For further results and different points of view on fractional calculus, see for instance [10–33].

## 3. Laplace's transform of fractional order

**Definition 3.1.** Let  $f(x)$  denote a function which vanishes for negative values of  $x$ . Its Laplace's transform  $L_\alpha \{f(x)\}$  of order  $\alpha$  (or its  $\alpha$ -th fractional Laplace's transform) is defined by the following expression, when it is finite:

$$L_\alpha \{f(x)\} := F_\alpha(s) = \int_0^\infty E_\alpha(-s^\alpha x^\alpha) f(x)(dx)^\alpha, \quad (3.1)$$

$$:= \lim_{M \uparrow \infty} \int_0^M E_\alpha(-s^\alpha x^\alpha) f(x)(dx)^\alpha, \quad (3.2)$$

where  $s \in \mathbb{C}$ , and  $E_\alpha(u)$  is the Mittag-Leffler function  $\sum u^k/(k\alpha)!$ .

The following operational formulae can be easily obtained:

$$L_\alpha \{x^\alpha f(x)\} = -D_s^\alpha L_\alpha \{f(x)\}, \quad (3.3)$$

$$L_\alpha \{f(ax)\}_s = (1/a)^\alpha L_\alpha \{f(x)\}_{s/a}, \quad (3.4)$$

$$L_\alpha \{f(x-b)\} = E_\alpha(-s^\alpha b^\alpha) L_\alpha \{f(x)\}, \quad (3.5)$$

$$L_\alpha \{E_\alpha(-c^\alpha x^\alpha) f(x)\}_s = L_\alpha \{f(x)\}_{s+c}. \quad (3.6)$$

$$L_\alpha \{-x^\alpha f(x)\} = D_s^\alpha L_\alpha \{f(x)\}, \quad (3.7)$$

$$L_\alpha \left\{ \int_0^x f(u)(du)^\alpha \right\} = \Gamma^{-1}(1+\alpha) s^{-\alpha} L_\alpha \{f(x)\}. \quad (3.8)$$

Furthermore, using the properties of the Mittag-Leffler function and integration by parts, we find that

$$L_\alpha \{f^{(\alpha)}(x)\} = s^\alpha L_\alpha \{f(x)\} - \Gamma(1+\alpha)f(0). \quad (3.9)$$

**Proof of (3.3).** We differentiate the expression for  $L_\alpha \{f(x)\}$  with respect to  $s$ , and notice that

$$D_s^\alpha E_\alpha(-s^\alpha x^\alpha) = -x^\alpha E_\alpha(-s^\alpha x^\alpha). \quad \blacksquare$$

**Proof of (3.4).** We start from the equality

$$\int_0^M E_\alpha(-s^\alpha x^\alpha) f(ax)(dx)^\alpha = \alpha \int_0^M (M-x)^{\alpha-1} E_\alpha(-s^\alpha x^\alpha) f(ax) dx,$$

and on making the change of variable  $ax \leftarrow u$  in the right-side term, we obtain the equality

$$\int_0^{aM} \left(M - \frac{u}{a}\right)^{\alpha-1} E_\alpha\left(-s^\alpha \frac{u^\alpha}{a^\alpha}\right) f(u) \frac{du}{a} = \int_0^{aM} \frac{(aM-u)^{\alpha-1}}{a^{\alpha-1}} E_\alpha\left(-s^\alpha \frac{u^\alpha}{a^\alpha}\right) f(u) \frac{du}{a}. \quad \blacksquare$$

**Proof of (3.5).** Using the change of variable  $x-b \leftarrow u$ , one has the equality

$$\int_0^M (M-x)^{\alpha-1} E_\alpha(-s^\alpha x^\alpha) f(x-b) dx = \int_0^{M-b} (M-b-u)^{\alpha-1} E_\alpha(-s^\alpha (b+u)^\alpha) f(u) du$$

and we take account of the equality

$$E_{\alpha}(\lambda(x+y)^{\alpha}) = E_{\alpha}(\lambda x^{\alpha}) E_{\alpha}(\lambda y^{\alpha}). \quad \blacksquare \quad (3.10)$$

**Proof of (3.6).** It is sufficient to note that one has the equality (see (3.7) above)

$$\int_0^{\infty} E_{\alpha}(-s^{\alpha} x^{\alpha}) E_{\alpha}(-c^{\alpha} x^{\alpha}) f(x) (dx)^{\alpha} = \int_0^{\infty} E_{\alpha}(-x^{\alpha} (s+c)^{\alpha}) f(x) (dx)^{\alpha}. \quad \blacksquare$$

**Proof of (3.7).** Directly by taking the fractional derivative of order  $\alpha$  of  $L_{\alpha}\{f(x)\}$  in (3.1), with respect to  $s$ .  $\blacksquare$

**Proof of (3.8).** One starts from the equality

$$D_x^{\alpha} \int_0^x f(u) (du)^{\alpha} = \alpha! f(x).$$

Combining with (3.8) (which we shall prove shortly), we can write

$$\begin{aligned} s^{\alpha} L_{\alpha} \left\{ D_x^{\alpha} \int_0^x f(u) (du)^{\alpha} \right\} &= s^{\alpha} L_{\alpha} \left\{ \int_0^x f(u) (du)^{\alpha} \right\} \\ &= \alpha! L_{\alpha} \{f(x)\}. \quad \blacksquare \end{aligned}$$

**Proof of (3.9).** Using the fact that

$$D_x^{\alpha} E_{\alpha}(-s^{\alpha} x^{\alpha}) = -s^{\alpha} E_{\alpha}(-s^{\alpha} x^{\alpha}),$$

the equality

$$(uv)^{(\alpha)} = u^{(\alpha)} v + uv^{(\alpha)}$$

yields

$$D_x^{\alpha} (E_{\alpha}(-s^{\alpha} x^{\alpha}) f(x)) = -s^{\alpha} E_{\alpha}(-s^{\alpha} x^{\alpha}) f(x) + E_{\alpha}(-s^{\alpha} x^{\alpha}) f^{(\alpha)}(x).$$

Integrating both sides with respect to  $x$  and using the equality

$$D^{-\alpha} f = (\alpha!)^{-1} \int_0^x f(\xi) (d\xi)^{\alpha}$$

yields the result.  $\blacksquare$

#### 4. The convolution theorem for the fractional Laplace's transform

**Proposition 4.1.** If we define the convolution of order  $\alpha$  of the two functions  $f(x)$  by the expression

$$(f(x) * g(x))_{\alpha} := \int_0^x f(x-u) g(u) (du)^{\alpha}, \quad (4.1)$$

then one has the equality

$$L_{\alpha} \{ (f(x) * g(x))_{\alpha} \} = L_{\alpha} \{f(x)\} L_{\alpha} \{g(x)\}. \quad (4.2)$$

**Proof.** One starts from the definition

$$\begin{aligned} L_{\alpha} \{ (f * g)_{\alpha} \} &= \int_0^{\infty} (dx)^{\alpha} E_{\alpha}(-s^{\alpha} x^{\alpha}) \int_0^x f(x-u) g(u) (du)^{\alpha} \\ &= \int_0^{\infty} (dx)^{\alpha} E_{\alpha}(-s^{\alpha} (x-u)^{\alpha}) E_{\alpha}(-s^{\alpha} u^{\alpha}) \int_0^x f(x-u) g(u) (du)^{\alpha}. \end{aligned}$$

This being the case, we make the change of variable  $y = x - u$ ,  $v = u$  to obtain

$$L_{\alpha} \{ (f * g)_{\alpha} \} = \int_0^{\infty} \int_0^{\infty} E_{\alpha}(-s^{\alpha} y^{\alpha}) E_{\alpha}(-s^{\alpha} v^{\alpha}) f(y) g(v) (dy)^{\alpha} (dv)^{\alpha}. \quad \blacksquare$$

## 5. Inversion formula for the fractional Laplace's transform

### 5.1. Dirac's delta distribution (or generalized function) of fractional order

**Definition 5.1.** Dirac's distribution, or generalized function,  $\delta_\alpha(x)$  of fractional order  $\alpha$ ,  $0 < \alpha < 1$ , is defined by the equality

$$\int_{\mathfrak{R}} f(x) \delta_\alpha(x) (dx)^\alpha = \alpha f(0). \quad (5.1)$$

**Lemma 5.1.** Define the function

$$\delta_\alpha(x, \varepsilon) = \begin{cases} 0, & x \notin [0, \varepsilon] \\ \varepsilon^{-\alpha}, & 0 < x \leq \varepsilon \end{cases}; \quad (5.2)$$

then one has the limit

$$\lim_{\varepsilon \downarrow 0} \delta_\alpha(x; \varepsilon) = \delta_\alpha(x). \quad (5.3)$$

**Proof.** It is sufficient to calculate the left side of (5.1). ■

### 5.2. Fractional Dirac's delta generalized function and the Mittag–Leffler function

The relation between  $\delta_\alpha(x)$  and  $E_\alpha(x^\alpha)$  is clarified by the following result which will be of basic help later when we deal with Fourier's transform of fractional order.

**Lemma 5.2.** The following equality holds:

$$\frac{\alpha}{(M_\alpha)^\alpha} \int_{-\infty}^{+\infty} E_\alpha(i(-\omega x)^\alpha) (d\omega)^\alpha = \delta_\alpha(x), \quad (5.4)$$

where  $M_\alpha$  is the period of the complex-valued Mittag–Leffler function defined by the equality  $E_\alpha(i(M_\alpha)^\alpha) = 1$ .

**Proof.** We check that (5.4) is consistent with

$$\alpha = \int_{\mathfrak{R}} E_\alpha(i\omega^\alpha x^\alpha) \delta_\alpha(x) (dx)^\alpha,$$

and to this end, we replace  $\delta_\alpha(x)$  in this expression by (5.2) to obtain

$$\begin{aligned} \alpha &= \int_{\mathfrak{R}} (dx)^\alpha \int_{\mathfrak{R}} \frac{\alpha}{(M_\alpha)^\alpha} E_\alpha(ix^\alpha(\omega - u)^\alpha) (du)^\alpha \\ &= \int_{\mathfrak{R}} (dx)^\alpha \int_{\mathfrak{R}} \frac{\alpha}{(M_\alpha)^\alpha} E_\alpha(i(-vx)^\alpha) (dv)^\alpha \\ &= \int_{\mathfrak{R}} \delta_\alpha(x) (dx)^\alpha. \quad \blacksquare \end{aligned}$$

### 5.3. Laplace's transform inversion theorem

**Proposition 5.1.** Given the Laplace's transform (3.1) that we recall here for convenience:

$$F_\alpha(s) := \int_0^{+\infty} E_\alpha(-s^\alpha x^\alpha) f(x) (dx)^\alpha, \quad 0 < \alpha < 1, \quad (5.5)$$

one has the inversion formula

$$f(x) = \frac{1}{(M_\alpha)^\alpha} \int_{-i\infty}^{+i\infty} E_\alpha(s^\alpha x^\alpha) F_\alpha(s) (ds)^\alpha. \quad (5.6)$$

**Proof.** Substituting (5.5) into (5.6), we have successively

$$\begin{aligned} f(x) &= \frac{1}{(M_\alpha)^\alpha} \int_{-i\infty}^{+i\infty} (dx)^\alpha E_\alpha(s^\alpha x^\alpha) \int_0^\infty E_\alpha(-s^\alpha u^\alpha) f(u) (du)^\alpha \\ &= \frac{1}{(M_\alpha)^\alpha} \int_0^\infty f(u) (du)^\alpha \int_{-i\infty}^{+i\infty} E_\alpha(s^\alpha (x-u)^\alpha) (ds)^\alpha \\ &= \frac{1}{\alpha} \int_{\Re} f(u) \delta_\alpha(x-u) (du)^\alpha. \quad \blacksquare \end{aligned}$$

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